

GRADED MORITA EQUIVALENCE OF CLIFFORD SUPERALGEBRAS

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ABSTRACT. This note uses a variation of graded Morita theory for finite dimensional superalgebras to determine explicitly the graded basic superalgebras for all real and complex Clifford superalgebras. As an application, the Grothendieck groups of the category of left \mathbb{Z}_2 -graded modules over all real and complex Clifford superalgebras are described explicitly.

1. INTRODUCTION

Clifford (super)algebras or special Grassmann (super)algebras play an important role in many branches of mathematics such as Clifford analysis, algebras, mathematics physics, geometry and topology etc., see for example [1, 12]. For the finite dimensional superalgebras, there is a natural graded Morita theory, which states that a finite dimensional superalgebra is graded Morita equivalent to a graded basic superalgebra, see [6, Theorem 2.3] for a more general context.

Note that some important algebraic objects such as the Hochschild and cyclic (co)homology of superalgebras are graded Morita invariants [15] and that the fundamental relationship between Clifford (super)algebras and Bott periodicity is established by Atiyah, Bott and Shapiro [1], see also [8] for an exposition. It is a natural and interesting problem to determine explicitly the graded basic superalgebras for Clifford superalgebras. In this note, using a variation of the graded Morita equivalent theory for finite dimensional superalgebras (Proposition-Definition 3.8), we determine explicitly the graded basic superalgebras for all real and complex Clifford superalgebras (Theorems 4.8 and 4.12). As an application, we determine explicitly the Grothendieck groups of \mathbb{Z}_2 -graded modules over all real and complex Clifford superalgebras (Theorem 5.4), which is useful to our understanding of the Atiyah-Bott-Shapiro isomorphisms.

The lay-out of this note as follows. We begin in Section 2 with preliminaries on superalgebras and fix our notations. In section 3 we review briefly the graded Morita theory for superalgebras and give some useful lemmas. The graded basic superalgebras for all real and complex Clifford superalgebras are determined explicitly in section 4. Finally, as an application, we determine explicitly the Grothendieck groups of \mathbb{Z}_2 -graded modules over all real and complex Clifford superalgebras in section 5.

2. PRELIMINARIES ON SUPERALGEBRAS

In this section, we recall some facts on superalgebras and fix the notations, our references are [2], [9, Chapter 3] and [7, §1].

2.1. Let K be a field and $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. By a *superspace* we mean a \mathbb{Z}_2 -graded K -vector space V , namely a K -vector space with a decomposition into two subspaces $V = V_0 \oplus V_1$. A nonzero element v of V_i will be called *homogeneous* and we denote its degree by $|v| = i \in \mathbb{Z}_2$. We will view K as a superspace concentrated in degree 0.

Given superspaces V and W , we view the direct sum $V \oplus W$ and the tensor product $V \otimes_K W$ as superspaces with $(V \oplus W)_i = V_i \oplus W_i$, and $(V \otimes_K W)_i = V_0 \otimes_K W_i \oplus V_1 \otimes_K W_{1-i}$ for $i \in \mathbb{Z}_2$. With this grading, $V \otimes_K W$ is called the *skew tensor product* of V and W and is denoted by $V \hat{\otimes}_K W$. Also, we make the vector space $\text{Hom}_K(V, W)$ of all K -linear maps from V to W into a superspace by setting that $\text{Hom}_K(V, W)_i$ consists of all the K -linear maps $f : V \rightarrow W$ with $f(V_j) \subseteq W_{i+j}$ for $i, j \in \mathbb{Z}_2$. Elements of $\text{Hom}_K(V, W)_0$ (resp. $\text{Hom}_K(V, W)_1$) will be referred to as *even* (resp. *odd*) *linear maps*.

2.2. Recall that a *superalgebra* A is both a superspace and an associative K -algebra with identity such that $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$. By a superalgebra *homomorphism* we mean an even linear map which is an algebra homomorphism in the usual sense and write $A \cong B$ provided that the superalgebras A and B are isomorphic.

Given two superalgebras A and B , the *skew tensor product* $A \hat{\otimes}_K B$ is again a superalgebra with the inducing grading and multiplication given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2, \text{ for } a_i \in A \text{ and } b_i \in B.$$

2010 *Mathematics Subject Classification.* Primary 15A66, 15A75; Secondary 16G99, 19A99.

Key words and phrases. Clifford superalgebras; graded Morita equivalence; graded basic superalgebras.

The author is supported in part by NSFC (No. 11001253 and No. 11101037).

Note this and other such expressions only make sense for homogeneous elements. Observe that the skew tensor product $A \widehat{\otimes}_K A \widehat{\otimes}_K \cdots \widehat{\otimes}_K A$ (n factors) is well-defined and that the *supertwist map* $T_{A,B} : A \widehat{\otimes}_K B \rightarrow B \widehat{\otimes}_K A$, $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$, for $a \in A, b \in B$, is an isomorphism of superalgebras.

2.3. The main principle for superalgebras is the following *rule of signs* [5, §III.4] or [9, §3.1.1]: if in some formula of usual algebra there are monomials with interchanged terms, then in the corresponding formula in superalgebra every interchange neighboring terms, say a and b , is accompanied by the multiplication of the monomial by the factor $(-1)^{|a||b|}$; or equivalently, each letter a_i appearing as an argument on the left-hand side (LHS) of the defining equation has a degree associated with it, and a factor of (-1) is introduced on the right-hand side (RHS) each time a pair of letters on the LHS, both of odd degree, appearing in reverse order on the RHS. In order for this rule make sense it is essential that every letter on the LHS should appear exactly once on the RHS.

2.4. Let A be a superalgebra. By a graded A -module M we mean a \mathbb{Z}_2 -graded left A -module, that is, M is both a superspace $M = M_0 \oplus M_1$ and a left A -module such that $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$. We denote by \widehat{A} the superalgebra with the same underlying superspace as A but new multiplication $\widehat{a}\widehat{b} = (-1)^{|a||b|} ab$. If M is a graded A -module, let \widehat{M} denote the \widehat{A} -module with M as the underlying superspace and operators defined as $\widehat{a}\widehat{m} = (-1)^{|a||m|} a\widehat{m}$ for $a \in A, m \in M$.

Unless otherwise explicitly stated, all our modules will be \mathbb{Z}_2 -graded left modules. We denote by $\text{Mod} A$ the category of all A -modules with morphisms

$$\text{Hom}_{\text{Mod} A}(M, N) := \text{Hom}_{\text{Mod} A}(M, N)_0 + \text{Hom}_{\text{Mod} A}(M, N)_1,$$

where $\text{Hom}_{\text{Mod} A}(M, N)_i$, $i \in \mathbb{Z}_2$, is consisting of all K -linear maps f from M to N such that $f(M_j) \subseteq N_{i+j}$ and $f(am) = (-1)^{i|a|} af(m)$, for all $a \in A, m \in M$ and $j \in \mathbb{Z}_2$. The elements of $\text{Hom}_{\text{Mod} A}(M, N)_0$ (resp. $\text{Hom}_{\text{Mod} A}(M, N)_1$) are called *even* (resp. *odd*) *homomorphisms* from M to N . Let $\text{Gr} A$ be the category of all A -modules with even homomorphisms.

3. GRADED MORITA EQUIVALENT THEORY

3.1. **Definition.** Let A be a superalgebra. The *parity change* (reps. *suspension*) functor π (resp. σ) from $\text{Gr} A$ to itself is defined as following: for $M = M_0 \oplus M_1$ in $\text{Gr} A$, we define the graded A -module $\pi(M)$ (resp. $\sigma(M)$) by the conditions:

- (i) $\pi(M)$ and $\sigma(M)$ are superspaces with $\pi(M)_i = \sigma(M)_i = M_{i+1}$ for $i \in \mathbb{Z}_2$;
- (ii) the module structure on $\pi(M)$ (resp. $\sigma(M)$) is defined by $a\pi(m) := (-1)^{|a|}\pi(am)$ (resp. $a\sigma(m) := \sigma(am)$) for homogeneous elements $a \in A$ and $m \in M$.

For a homomorphism f , we define $\pi(f)$ (resp. $\sigma(f)$) is the same underlying linear map as f .

3.2. *Remark.* It follows by definition that $\pi^2 = \text{Id}_{\text{Gr} A}$ and $\sigma^2 = \text{Id}_{\text{Gr} A}$, which means that the parity change functor π and the suspension functor σ are shifts of the category $\text{Gr} A$ in the sense of [13, Definition 3.2] and [11, §2].

From now on we assume that S is either the parity change functor π or the suspension functor σ , unless otherwise explicitly stated. For an A -module M , following [11, §2], we define the *S -twisted endomorphism superalgebra* of M to be the superalgebra

$$\text{End}_A^S(M) := \text{Hom}_{\text{Gr} A}(M, M) \oplus \text{Hom}_{\text{Gr} A}(S(M), M),$$

and define the *S -twisted Hom functor* to be the functor

$$\text{Hom}_A^S(M, -) : \text{Gr} A \rightarrow \text{Gr} \text{End}_A^S(M), \quad N \mapsto \text{Hom}_{\text{Gr} A}(M, N) \oplus \text{Hom}_{\text{Gr} A}(S(M), N).$$

Denote by $\text{Mod}^S A$ the category of all graded A -modules with homomorphisms $\text{Hom}_A^S(M, N)$.

3.3. **Lemma** (cf. [2], p. 123). *Assume that A is a finite dimensional superalgebras and that M and N are A -modules. Then $\text{Hom}_{\text{Mod}^\pi A}(M, N) = \text{Hom}_{\text{Mod}^\sigma \widehat{A}}(\widehat{M}, \widehat{N})$ and $\text{Mod}^\pi A = \text{Mod} A = \text{Mod}^\sigma \widehat{A}$.*

Proof. First note that for A -modules M and N , by definition,

$$\text{Hom}_{\text{Mod}^\pi A}(M, N) = \text{Hom}_A^\pi(M, N) = \text{Hom}_{\text{Gr} A}(M, N) \oplus \text{Hom}_{\text{Gr} A}(\pi(M), N) \text{ and}$$

$$\mathrm{Hom}_{\mathrm{Mod}^\sigma \widehat{A}}(\widehat{M}, \widehat{N}) = \mathrm{Hom}_{\widehat{A}}^\sigma(\widehat{M}, \widehat{N}) = \mathrm{Hom}_{\mathrm{Gr} \widehat{A}}(\widehat{M}, \widehat{N}) \oplus \mathrm{Hom}_{\mathrm{Gr} \widehat{A}}(\sigma(\widehat{M}), \widehat{N}).$$

Secondly by definition, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Gr} \widehat{A}}(\widehat{M}, \widehat{N}) &= \{f : \widehat{M} \rightarrow \widehat{N} \mid f(\widehat{M}_i) \subseteq \widehat{N}_i \text{ and } f(\hat{a}\hat{m}) = \hat{a}f(\hat{m}), \forall i \in \mathbb{Z}_2, a \in A, m \in M\} \\ &= \{f : M \rightarrow N \mid f(M_i) \subseteq N_i \text{ and } f(am) = af(m), \forall i \in \mathbb{Z}_2, a \in A, m \in M\} \\ &= \mathrm{Hom}_{\mathrm{Gr} A}(M, N); \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Gr} \widehat{A}}(\sigma(\widehat{M}), \widehat{N}) &= \{f : \sigma(\widehat{M}) \rightarrow \widehat{N} \mid f(\sigma(\widehat{M})_i) \subseteq \widehat{N}_i \text{ and } f(\hat{a}\sigma(\hat{m})) = \hat{a}f(\sigma(\hat{m})), \forall i \in \mathbb{Z}_2, a \in A, m \in M\} \\ &= \{f : M \rightarrow N \mid f(M_i) \subseteq N_{1+i} \text{ and } f(am) = (-1)^{|a|}af(m), \forall i \in \mathbb{Z}_2, a \in A, m \in M\} \\ &= \mathrm{Hom}_{\mathrm{Gr} A}(\pi(M), N). \end{aligned}$$

Therefore $\mathrm{Hom}_{\mathrm{Mod}^\pi A}(M, N) = \mathrm{Hom}_{\mathrm{Mod}^\sigma \widehat{A}}(\widehat{M}, \widehat{N})$.

Finally note that the objects of the categories $\mathrm{Mod}^\pi A$, $\mathrm{Mod} A$ and $\mathrm{Mod}^\sigma \widehat{A}$ are same. By the first part of the lemma, we only need to show that $\mathrm{Hom}_{\mathrm{Mod}^\pi A}(M, N) = \mathrm{Hom}_{\mathrm{Mod} A}(M, N)$ for all A -modules M and N ; furthermore, it suffices to show that $\mathrm{Hom}_{\mathrm{Gr} A}(\pi(M), N) = \mathrm{Hom}_{\mathrm{Mod} A}(M, N)_1$ for all A -modules M and N . Indeed for all A -modules M and N , by definition,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Gr} A}(\pi(M), N) &= \{f : \pi(M) \rightarrow N \mid f(\pi(M)_i) \subseteq N_i \text{ and } f(a\pi(m)) = af(\pi(m)), \forall i \in \mathbb{Z}_2, a \in A, m \in M\} \\ &= \{f : M \rightarrow N \mid f(M_{i+1}) \subseteq N_i \text{ and } f(\pi(am)) = (-1)^{|a|}af(\pi(m)), \forall i \in \mathbb{Z}_2, a \in A, m \in M\} \\ &= \{f : M \rightarrow N \mid f(M_i) \subseteq N_{1+i} \text{ and } f(am) = (-1)^{|a|}af(m), \forall i \in \mathbb{Z}_2, a \in A, m \in M\} \\ &= \mathrm{Hom}_{\mathrm{Mod} A}(M, N)_1, \end{aligned}$$

where the second and third equalities follow by Definition 3.1 that $\pi(M)_i = M_{i+1}$, $a\pi(m) = (-1)^{|a|}\pi(am)$, and that $f(\pi(am)) = f(am)$ for homogeneous elements $a \in A$, $m \in M$ and homogeneous homomorphisms f . As a consequence, we complete the proof. \square

We say that a non-zero A module M is *gr-indecomposable* if it is not the direct sum of two non-zero modules that M is *S-indecomposable* if it is not the direct sum of two non-zero A -modules in the category $\mathrm{Mod}^S A$. A superalgebra is called to be *gr-divisional* (resp. *gr-local*) if every nonzero homogeneous element of it is invertible (resp. either invertible or nilpotent).

3.4. Lemma ([6], §2.2). *Let A be a finite dimensional superalgebra over K and $M \in \mathrm{Gr} A$. Then*

- (i) *M is S-indecomposable if and only if M is gr-indecomposable.*
- (ii) *M is gr-indecomposable if and only if $\mathrm{End}_A^S(M)$ is gr-local.*
- (iii) *If M is gr-simple then $\mathrm{End}_A^S(M)$ is either a gr-divisional superalgebra concentrated in degree zero, or a gr-divisional superalgebra containing an odd involuting element.*

3.5. Definition. Let A and B be two finite dimensional superalgebras over K . We say that A and B are *S-graded equivalent*, denote by $A \approx_S B$, if the categories $\mathrm{Gr} A$ and $\mathrm{Gr} B$ are equivalent and $B \cong \mathrm{End}_A^S(P)$ for some finitely generated projective A -module P .

The relationship between π -graded equivalence and σ -graded equivalence is the following:

3.6. Lemma. *Assume that finite dimensional superalgebras A and B are σ -graded equivalent. Then A and \widehat{B} are π -graded equivalent.*

Proof. Assume that the categories $\mathrm{Gr} A$ and $\mathrm{Gr} B$ are equivalent and that there is a finitely generated projective A -module P such that $B \cong \mathrm{End}_A^\sigma(P)$. Observe that superalgebras B and $\widehat{\widehat{B}}$ are isomorphic and that $\mathrm{Gr} B$ and $\mathrm{Gr} \widehat{B}$ are equivalent. Applying Lemma 3.3, $\mathrm{End}_A^\pi(\widehat{P}) = \mathrm{End}_{\widehat{A}}^\sigma(\widehat{P}) = \mathrm{End}_A^\sigma(P)$. The proof is completed. \square

Let A be a finite dimensional superalgebra. Then A has a complete set of primitive orthogonal idempotents and denoted it by $\{f_1, \dots, f_n\}$. Let $P_i = Af_i$ be the projective A -module corresponding

to the primitive orthogonal idempotent f_i of A . Then $\widehat{P}_i = \widehat{A}f_i$ and Lemma 3.3 and [4, Corollary 3.10] imply that $\text{Hom}_{\text{Mod}^\pi A}(P_i, P_j) = \text{Hom}_{\text{Mod}^\sigma A}(\widehat{P}_i, \widehat{P}_j) = f_i \widehat{A} f_j$ and $\text{Hom}_{\text{Mod}^\sigma A}(P_i, P_j) = f_i A f_j$.

We say that two idempotents f and g of A are S -equivalent if the A -modules Af and Ag are isomorphic in the category $\text{Mod}^S A$. The following is a criterion of S -equivalent of idempotents.

3.7. Lemma. *The idempotents f and g are S -equivalent if and only if there exist homogeneous elements $x \in fAg$ and $y \in gAf$ satisfying $xgy = f$ and $yfx = g$.*

Proof. Note that $\text{Hom}_{\text{Mod}^S A}(Af, Ag)$ is either fAg or $f\widehat{A}g$ and that $\text{Hom}_{\text{Mod}^S A}(Ag, Af)$ is either gAf or $g\widehat{A}f$. Assume that Af and Ag are isomorphic in the category $\text{Mod}^S A$. Then there exists invertible homogenous homomorphisms induced by homogeneous elements $x \in fAg$ and $y \in gAf$ satisfying $xgy = f$ and $yfx = g$. Conversely, suppose that there exist homogeneous elements $x \in fAg$ and $y \in gAf$ satisfying $xgy = f$ and $yfx = g$. Then x and y induce natural homomorphisms $Af \rightarrow Ag$, $f \mapsto xg$ and $Ag \rightarrow Af$, $g \mapsto yf$ respectively, which give the desired isomorphisms. \square

The following definition is our investigation for all Clifford superalgebras in this note.

3.8. Proposition-Definition (cf. Proposition 2.4, [6]). *Let A be a finite dimensional superalgebra and $J(A)$ the graded Jacobson radical of A . We say that the superalgebra A is **S-graded basic** if it satisfies the following equivalent conditions:*

- (i) f_i are pairwise non- S -equivalent for $1 \leq i \leq n$.
- (ii) Any decomposition of A into indecomposable projective A -modules $A = \bigoplus_{i=1}^n P_i$ satisfies P_i and P_j are not isomorphic in $\text{Mod}^S A$ for all $1 \leq i \neq j \leq n$.
- (iii) $A/J(A)$ is a direct sum of gr-divisional superalgebras.

3.9. Remark. By [6, Theorem 2.3], a finite dimensional superalgebra A is S -graded equivalent to an S -graded basic superalgebra. More precisely, let $\{e_i \mid 1 \leq i \leq n\}$ be a complete set of non- S -equivalent orthogonal primitive idempotents of A . It follows that A is S -graded equivalent to $\text{End}_A^S(\bigoplus_{i=1}^n Ae_i)$, which is S -graded basic.

The following Lemma is the key to the graded Morita classification for Clifford superalgebras.

3.10. Lemma. *Suppose that the superalgebras A and A' are S -graded equivalent to B and B' respectively. Then $A \widehat{\otimes}_K A'$ is S -graded equivalent to $B \widehat{\otimes}_K B'$.*

Proof. Without loss of generality, we may assume that both B and B' are S -graded basic and that $\{f_1, \dots, f_r\}$ and $\{f'_1, \dots, f'_{r'}\}$ are the complete sets of orthogonal primitive idempotents of B and B' respectively. Now let $\{e_1, \dots, e_n\}$ and $\{e'_1, \dots, e'_{n'}\}$ be the complete sets of orthogonal primitive idempotents of A and A' respectively. Then $\{f_1, \dots, f_r\} \subseteq \{e_1, \dots, e_n\}$ and $\{f'_1, \dots, f'_{r'}\} \subseteq \{e'_1, \dots, e'_{n'}\}$. Observe that $\{e_i \otimes e'_j \mid 1 \leq i \leq n, 1 \leq j \leq n'\}$ and $\{f_i \otimes f'_j \mid 1 \leq i \leq r, 1 \leq j \leq r'\}$ is a set of orthogonal idempotents of $A \widehat{\otimes}_K A'$ such that $\sum_{i,j} e_i \otimes e'_j = 1_A \otimes 1_{A'}$, and $\{f_i \otimes f'_j \mid 1 \leq i \leq r, 1 \leq j \leq r'\}$ is a set of orthogonal idempotents of $B \widehat{\otimes}_K B'$ such that $\sum_{i,j} f_i \otimes f'_j = 1_B \otimes 1_{B'}$. Note that $\{f_i \otimes f'_j \mid 1 \leq i \leq r, 1 \leq j \leq r'\} \subseteq \{e_i \otimes e'_j \mid 1 \leq i \leq n, 1 \leq j \leq n'\}$, which implies that the complete set of non- S -equivalent orthogonal primitive idempotents of $A \widehat{\otimes}_K A'$ can be obtained from $\{f_i \otimes f'_j \mid 1 \leq i \leq r, 1 \leq j \leq r'\}$ by decomposing these orthogonal idempotents. As a consequence, $A \widehat{\otimes}_K A'$ is S -graded equivalent to $B \widehat{\otimes}_K B'$. \square

We close this section with some remarks.

- 3.11. Remark.**
- (i) The σ -graded equivalence is exactly the so-called *graded Morita equivalence* defined in [13, 11] for group-graded algebras by ignoring the rule of signs in the case of superalgebras.
 - (ii) A finite dimensional superalgebra is σ -graded equivalent but not π -graded equivalent to itself.
 - (iii) The rule of signs implies that for finite dimensional superalgebra the π -graded equivalence is deserving of a better understanding than σ -graded equivalence, that is, for finite dimensional superalgebra A the category $\text{Mod} A$ should be study exhaustively.

- (iv) Lemmas 3.6 and 3.7 show that the S -graded basic classification of finite dimensional superalgebras can be determined completely by its graded Morita equivalent classification, that is, by its σ -graded equivalent classification.

4. GRADED MORITA EQUIVALENT CLASSIFICATION OF CLIFFORD SUPERALGEBRAS

From now on we will write \mathbb{R} , \mathbb{C} and \mathbb{H} respectively for the real, complex and quaternion number-fields and view them as superalgebras over \mathbb{R} concentrated on degree zero respectively and we will say a superalgebra to be graded basic if it is either σ -graded basic or π -graded basic (cf. Remark 3.11(iv)). Note that \mathbb{R} , \mathbb{C} and \mathbb{H} are gr-divisional superalgebras; in particular, they are graded basic superalgebras according to Proposition-Definition 3.8.

4.1. Let p, q and r be positive integers. Following Porteous [10], we denote by $\mathbb{R}^{p,q,r}$ the real quadratic space \mathbb{R}^{p+q+r} with the quadratic form $\rho(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$. The Clifford superalgebra $\mathbb{R}_{p,q,r}$ on $\mathbb{R}^{p,q,r}$ is the real unitary superalgebra generated by odd generators e_1, \dots, e_{p+q+r} subject to the following relations:

$$\begin{aligned} e_i e_j + e_j e_i &= 0 & \text{for } 1 \leq i \neq j \leq p+q+r; \\ e_i^2 &= -e_{j+p}^2 = 1 & \text{for } 1 \leq i \leq p \text{ and } 1 \leq j \leq q; \\ e_{p+q+k}^2 &= 0 & \text{for } 1 \leq k \leq r. \end{aligned}$$

Note that the Clifford superalgebra $\mathbb{R}_{0,0,r}$ is the real Grassmann superalgebra $\bigwedge_{\mathbb{R}}(r)$ generated by odd generators $e_{p+q+1}, \dots, e_{p+q+r}$. Observe that the orthogonal primitive idempotent of $\bigwedge_{\mathbb{R}}(r)$ is the unit 1 since $\bigwedge_{\mathbb{R}}(r)/J(\bigwedge_{\mathbb{R}}(r)) = \mathbb{R}$, which implies $\bigwedge_{\mathbb{R}}(r)$ are graded basic for all positive integer $r \geq 1$ according to Proposition-Definition 3.8.

For now on we write $\mathbb{R}_{p,q}$ for the Clifford superalgebra $\mathbb{R}_{p,q,0}$. It is well-known that there are superalgebras isomorphism $\mathbb{R}_{p,q,r} \cong \mathbb{R}_{p,q} \widehat{\otimes}_{\mathbb{R}} \bigwedge_{\mathbb{R}}(r)$ and $\mathbb{R}_{p,q} \widehat{\otimes}_{\mathbb{R}} \mathbb{R}_{p',q'} \cong \mathbb{R}_{p+p',q+q'}$ for all positive integers p, q, r, p' and q' , see [3] or [1, Proposition 1.6].

4.2. Let $V = \mathbb{R}^{p+q+r}$ and $\alpha : V \rightarrow V, v \mapsto -v$. Then α induces a automorphism of the (ungraded) Clifford algebra $\mathbf{Cl}(V, \rho) = T(V)/\langle v \otimes v - \rho(v)|v \in V \rangle$ where $T(V)$ is the tensor algebra of V . Since $\alpha^2 = \text{Id}$, there is a decomposition

$$\mathbf{Cl}(V, \rho) = \mathbf{Cl}^0(V, \rho) \oplus \mathbf{Cl}^1(V, \rho)$$

where $\mathbf{Cl}^i(V, \rho) = \{\phi \in \mathbf{Cl}(V, \rho) \mid \alpha(\phi) = (-1)^i \phi\}$ are the eigenspaces of α . Clearly, since $\alpha(\phi_1 \phi_2) = \alpha(\phi_1)\alpha(\phi_2)$, we have that

$$\mathbf{Cl}^i(V, \rho) \mathbf{Cl}^j(V, \rho) \subseteq \mathbf{Cl}^{i+j}(V, \rho) \quad \text{for all } i, j \in \mathbb{Z}_2,$$

that is, $\mathbf{Cl}(V, \rho)$ is a superalgebra. It is an observation of Atiyah, Bott and Shapiro [1] that this \mathbb{Z}_2 -grading plays an important role in the analysis and application of Clifford algebras.

4.3. *Remark.* The two superalgebras $\mathbb{R}_{p,q,r}$ and $\mathbf{Cl}(V, \rho)$ are naturally isomorphic.

Denote by \mathbb{D}_+ (resp. \mathbb{D}_-) the real superalgebra generated by odd element e_+ (resp. e_-) subject to the relation $e_+^2 = 1$ (resp. $e_-^2 = -1$). For a positive integer n , we denote by \mathbb{D}_+^n (resp. \mathbb{D}_-^n) the superalgebra $\mathbb{D}_+ \widehat{\otimes}_{\mathbb{R}} \cdots \widehat{\otimes}_{\mathbb{R}} \mathbb{D}_+$ (resp. $\mathbb{D}_- \widehat{\otimes}_{\mathbb{R}} \cdots \widehat{\otimes}_{\mathbb{R}} \mathbb{D}_-$) (n factors) and write $e_+^{i_1 \cdots i_n}$ (resp. $e_-^{i_1 \cdots i_n}$) for the homogeneous element $e_+^{i_1} \otimes \cdots \otimes e_+^{i_n} \in \mathbb{D}_+^n$ (resp. $e_-^{i_1} \otimes \cdots \otimes e_-^{i_n} \in \mathbb{D}_-^n$) where $i_j = 0, 1$ for all $1 \leq j \leq n$. Note that \mathbb{D}_+^n and \mathbb{D}_-^n are exactly $\mathbb{R}_{n,0}$ and $\mathbb{R}_{0,n}$ respectively.

4.4. **Lemma.** *If $n = 1, 2, 3$ then \mathbb{D}_+^n and \mathbb{D}_-^n are gr-divisional superalgebras.*

Proof. First, it is trivial that \mathbb{D}_{\pm} are gr-divisional superalgebras. Noticing that the even (resp. odd) elements of \mathbb{D}_+^2 are of the form $a \otimes 1 + be_+ \otimes e_+$ (resp. $a \otimes e_+ + be_+ \otimes 1$), where $a, b \in \mathbb{R}$, and that for all $a, b \in \mathbb{R}$,

$$(a \otimes 1 + be_+ \otimes e_+)(a \otimes 1 - be_+ \otimes e_+) = (a^2 + b^2) \otimes 1 = (a \otimes e_+ + be_+ \otimes 1)(a \otimes e_+ + be_+ \otimes 1).$$

Thus all non-zero homogeneous elements of \mathbb{D}_+^2 are invertible, that is, \mathbb{D}_+^2 is gr-divisional.

Set $\mathbf{1} := 1 \widehat{\otimes} 1 \widehat{\otimes} 1$, $\mathbf{i}_+ := e_+^{011}$, $\mathbf{j}_+ := e_+^{101}$, $\mathbf{k}_+ := e_+^{110}$ and $\theta_+ := e_+^{111}$. Then it follows directly that $\theta_+ \mathbf{i}_+ + \mathbf{i}_+ \theta_+ = 0$, $\theta_+ \mathbf{j}_+ + \mathbf{j}_+ \theta_+ = 0$ and $\theta_+ \mathbf{k}_+ + \mathbf{k}_+ \theta_+ = 0$. Furthermore, we have $\mathbb{R}\langle \mathbf{1}, \mathbf{i}_+, \mathbf{j}_+, \mathbf{k}_+ \rangle \cong \mathbb{H}$ and $\mathbb{D}_+^3 \cong \mathbb{H} \oplus \mathbb{H}\theta_+$. Note that θ_+ is a odd element of \mathbb{D}_+^3 satisfying $\theta_+^2 = -1$ and that \mathbb{H} is a gr-divisional superalgebra concentrated on degree zero. So \mathbb{D}_+^3 is gr-divisional.

Observe that the even (resp. odd) elements of \mathbb{D}_-^2 are of the form $a \otimes 1 + b e_- \otimes e_-$ (resp. $a \otimes e_- + b e_- \otimes 1$), where $a, b \in \mathbb{R}$, and that for all $a, b \in \mathbb{R}$,

$$(a \otimes 1 + b e_- \otimes e_-)(a \otimes 1 - b e_- \otimes e_-) = -(a^2 + b^2) \otimes 1 = (a \otimes e_- + b e_- \otimes 1)(a \otimes e_- + b e_- \otimes 1).$$

Thus all non-zero homogeneous elements of \mathbb{D}_-^2 are invertible, that is \mathbb{D}_-^2 is gr-divisional.

Set $\mathbf{i}_- := e_-^{011}$, $\mathbf{j}_- := e_-^{101}$, $\mathbf{k}_- := e_-^{110}$, and $\theta_- := e_-^{111}$. Then it follows directly that $\theta_- \mathbf{i}_- + \mathbf{i}_- \theta_- = 0$, $\theta_- \mathbf{j}_- + \mathbf{j}_- \theta_- = 0$, and $\theta_- \mathbf{k}_- + \mathbf{k}_- \theta_- = 0$. Moreover, we have $\mathbb{R}\langle \mathbf{1}, \mathbf{i}_-, \mathbf{j}_-, \mathbf{k}_- \rangle \cong \mathbb{H}$ and $\mathbb{D}_-^3 \cong \mathbb{H} \oplus \mathbb{H}\theta_-$. Note that θ_- is a odd element of \mathbb{D}_-^3 satisfying $\theta_-^2 = 1$ and that \mathbb{H} is a gr-divisional superalgebra concentrated on degree zero. Thus \mathbb{D}_-^3 is gr-divisional. \square

4.5. Lemma. $\mathbb{D}_+ \widehat{\otimes}_{\mathbb{R}} \mathbb{D}_-$ is graded Morita equivalent to the superalgebra \mathbb{R} .

Proof. Set $A = \mathbb{D}_+ \widehat{\otimes}_{\mathbb{R}} \mathbb{D}_-$. By Proposition-Definition 3.8, it suffices to determine the non- σ -equivalent idempotents of A . Let $f_{\pm} = \frac{1}{2}(1 \otimes 1 \pm e_+ \otimes e_-)$. Then $f_{\pm}^2 = f_{\pm}$, $f_+ + f_- = 1 \otimes 1$ and $f_+ f_- = 0 = f_- f_+$, which implies that $\{f_+, f_-\}$ is the complete set of orthogonal primitive idempotents of A since $\dim_{\mathbb{R}} A_0 = 2$. It is straightforward to show that

$$f_+ A f_- = \{r(e_+ \otimes 1 - 1 \otimes e_-) \mid r \in \mathbb{R}\} \quad \text{and} \quad f_- A f_+ = \{r(e_+ \otimes 1 + 1 \otimes e_-) \mid r \in \mathbb{R}\}.$$

Let $x = \frac{1}{2}(e_+ \otimes 1 - 1 \otimes e_-)$ and $y = \frac{1}{2}(e_+ \otimes 1 + 1 \otimes e_-)$. Then $x f_- y = f_+$ and $y f_+ x = f_-$. Applying Lemma 3.7, f_+ and f_- are σ -equivalent orthogonal primitive idempotents of A . As a consequence, Proposition-Definition 3.8 implies that A is graded Morita equivalent to $f_+ A f_+ = \mathbb{R} f_+ \cong \mathbb{R}$. \square

4.6. Lemma. \mathbb{D}_{\pm}^4 are graded Morita equivalent to the superalgebra \mathbb{H} .

Proof. By the proof of Lemma 4.4, the natural graded map $e_- \mapsto \theta_+$ gives an isomorphism between $\mathbb{D}_- \widehat{\otimes}_{\mathbb{R}} \mathbb{H}$ and \mathbb{D}_+^3 . Therefore, using Lemmas 3.10 and 4.5, $\mathbb{D}_+^4 = \mathbb{D}_+ \widehat{\otimes}_{\mathbb{R}} \mathbb{D}_+^3 \cong \mathbb{D}_+ \widehat{\otimes}_{\mathbb{R}} \mathbb{D}_- \widehat{\otimes}_{\mathbb{R}} \mathbb{H}$ is graded Morita equivalent to the superalgebra \mathbb{H} . Similarly, noticing that $\mathbb{D}_+ \widehat{\otimes}_{\mathbb{R}} \mathbb{H} \cong \mathbb{D}_-^3$, we can show that \mathbb{D}_-^4 is graded Morita equivalent to the superalgebra \mathbb{H} . \square

4.7. Remark. (i) The Lemmas 4.5 and 4.6 show that the skew tensor product of σ -graded basic superalgebras may not be σ -graded basic. In particular, the skew tensor product of gr-divisional superalgebras may not be gr-divisional.

(ii) Lemmas 3.6, 4.5 and 4.6 imply that $\mathbb{D}_+ \otimes \mathbb{D}_-$ and \mathbb{D}_{\pm}^4 are also π -graded equivalent to \mathbb{R} and \mathbb{H} respectively since $\widehat{\mathbb{D}}_{\pm} = \mathbb{D}_{\mp}$.

Now we can obtain the graded Morita classifications for all real Clifford superalgebras.

4.8. Theorem. Assume that p, q and r are non-negative integers. Then

$$\begin{aligned} \text{(i)} \quad \mathbb{R}_{p,q,r} &\approx_{\sigma} \begin{cases} \bigwedge_{\mathbb{R}}(r), & \text{if } p - q \equiv 0(\text{mod } 8); \\ \mathbb{H} \widehat{\otimes}_{\mathbb{R}} \bigwedge_{\mathbb{R}}(r), & \text{if } p - q \equiv 4(\text{mod } 8); \\ \mathbb{D}_+^i \widehat{\otimes}_{\mathbb{R}} \bigwedge_{\mathbb{R}}(r), & \text{if } p - q \equiv 4 - i(\text{mod } 8) \text{ for } 1 \leq i \leq 3; \\ \mathbb{D}_-^i \widehat{\otimes}_{\mathbb{R}} \bigwedge_{\mathbb{R}}(r), & \text{if } p - q \equiv 4 + i(\text{mod } 8) \text{ for } 1 \leq i \leq 3. \end{cases} \\ \text{(ii)} \quad \mathbb{R}_{p,q,r} &\approx_{\pi} \begin{cases} \bigwedge_{\mathbb{R}}(r), & \text{if } p - q \equiv 0(\text{mod } 8); \\ \mathbb{H} \widehat{\otimes}_{\mathbb{R}} \bigwedge_{\mathbb{R}}(r), & \text{if } p - q \equiv 4(\text{mod } 8); \\ \mathbb{D}_+^i \widehat{\otimes}_{\mathbb{R}} \bigwedge_{\mathbb{R}}(r), & \text{if } p - q \equiv 4 + i(\text{mod } 8) \text{ for } 1 \leq i \leq 3; \\ \mathbb{D}_-^i \widehat{\otimes}_{\mathbb{R}} \bigwedge_{\mathbb{R}}(r), & \text{if } p - q \equiv 4 - i(\text{mod } 8) \text{ for } 1 \leq i \leq 3. \end{cases} \end{aligned}$$

(Note that these superalgebras are graded basic superalgebras.)

Proof. (i) Note that $\mathbb{D}_+ \cong \mathbb{R}_{1,0}$ and $\mathbb{D}_- \cong \mathbb{R}_{1,0}$. Using [1, Proposition 1.6], Lemmas 3.10 and 4.5,

$$\mathbb{R}_{p,q} = \mathbb{D}_+^p \widehat{\otimes}_{\mathbb{R}} \mathbb{D}_-^q \approx_{\sigma} \begin{cases} \mathbb{D}_+^{p-q}, & \text{if } p \geq q; \\ \mathbb{D}_-^{q-p}, & \text{otherwise.} \end{cases}$$

Now by [10, §5.6], $\mathbb{H} \widehat{\otimes}_{\mathbb{R}} \mathbb{H}$ is isomorphic to the 4×4 real matrix superalgebra concentrated on degree zero, which is (graded) Morita equivalent to \mathbb{R} . Consequently, by Lemma 4.6, \mathbb{D}_{\pm}^8 is graded Morita equivalent to \mathbb{R} . Therefore Lemmas 3.10 and 4.6 imply that

$$\begin{aligned} \text{(i) if } p \geq q \text{ then } \mathbb{D}_+^{p-q} &\approx_{\sigma} \begin{cases} \mathbb{D}_+^i, & \text{if } p-q \equiv i \pmod{8}, i = 0, 1, 2, 3, \\ \mathbb{D}_+^i \widehat{\otimes}_{\mathbb{R}} \mathbb{H}, & \text{if } p-q \equiv 4+i \pmod{8}, i = 0, 1, 2, 3; \end{cases} \\ \text{(ii) if } p < q \text{ then } \mathbb{D}_-^{q-p} &\approx_{\sigma} \begin{cases} \mathbb{D}_-^i, & \text{if } q-p \equiv i \pmod{8}, i = 0, 1, 2, 3, \\ \mathbb{D}_-^i \widehat{\otimes}_{\mathbb{R}} \mathbb{H}, & \text{if } q-p \equiv 4+i \pmod{8}, i = 0, 1, 2, 3; \end{cases} \end{aligned}$$

By the proof of Lemma 4.6, $\mathbb{D}_+ \widehat{\otimes}_{\mathbb{R}} \mathbb{H} \cong \mathbb{D}_-^3$ and $\mathbb{D}_- \widehat{\otimes}_{\mathbb{R}} \mathbb{H} \cong \mathbb{D}_+^3$. Applying Lemmas 3.10 and 4.5 again, we have $\mathbb{D}_+^i \widehat{\otimes}_{\mathbb{R}} \mathbb{H} \approx_{\sigma} \mathbb{D}_-^{4-i}$ and $\mathbb{D}_-^i \widehat{\otimes}_{\mathbb{R}} \mathbb{H} \approx_{\sigma} \mathbb{D}_+^{4-i}$ for $1 \leq i \leq 3$. Noticing that $q-p \equiv 4+i \pmod{8}$ if and only if $p-q \equiv 4-i \pmod{8}$. As a consequence we complete the proof of (i) by [1, Proposition 1.6].

Note that $\widehat{\mathbb{R}_{p,q,r}} \cong \mathbb{R}_{q,p,r}$. Hence (ii) is follows directly by Lemma 3.6 and (i). \square

4.9. *Remark.* Note that the \mathbb{Z}_2 -graded Hochschild and cyclic (co)homology $H_*(A)$ of a finite dimensional superalgebra A is graded Morita equivalent invariant [15]. Theorem 4.8 implies that

$$H_*(\mathbb{R}_{p,q}) = \begin{cases} 0, & \text{if } * \geq 1; \\ \mathbb{R}, & \text{if } * = 0, \end{cases}$$

since \mathbb{R} , \mathbb{H} and \mathbb{D}_{\pm}^i for all $1 \leq i \leq 3$ are separable superalgebras (see [2, Chap. IV-V], which gives a slight generalization of [7, Proposition 1] since the quadratic form $\rho(x) = \sum_{i=1}^p x_i^2 - \sum_{j=1}^q x_{p+j}^2$ on \mathbb{R}^{p+q} is degenerate.

4.10. Let p and q be positive integers. We denote by $\mathbb{C}^{p,q}$ the complex quadratic space \mathbb{C}^{p+q} with a quadratic form $x_1^2 + \cdots + x_p^2$. The Clifford superalgebra $\mathbb{C}_{p,q}$ on $\mathbb{C}^{p,q}$ is the complex unitary superalgebra generated by odd generators e_1, \dots, e_{p+q} subject to the following relations:

$$\begin{aligned} e_i e_j + e_j e_i &= 0 & \text{for } 1 \leq i \neq j \leq p+q; \\ e_i^2 &= 1 & \text{for } 1 \leq i \leq p; \\ e_{p+j}^2 &= 0 & \text{for } 1 \leq j \leq q. \end{aligned}$$

Note that the Clifford superalgebra $\mathbb{C}_{0,q}$ is the complex Grassman superalgebra $\bigwedge_{\mathbb{C}}(q)$ generated by odd generators e_{p+1}, \dots, e_{p+q} . Observe that the orthogonal primitive idempotent of $\bigwedge_{\mathbb{C}}(q)$ is the unity 1, which implies that $\bigwedge_{\mathbb{C}}(q)$ are graded basic superalgebras for all positive integer $q \geq 1$.

We denote by $D = \mathbb{C} \oplus \mathbb{C}\varepsilon$ the complex superalgebra generated by odd generators ε subject to the relation $\varepsilon^2 = 1$. For positive integer n , let D^n be the superalgebra $D \widehat{\otimes}_{\mathbb{C}} \cdots \widehat{\otimes}_{\mathbb{C}} D$ (n factors). Then $D^p \cong \mathbb{C}_{p,0}$ and $\mathbb{C}_{p,q} \cong D^p \widehat{\otimes}_{\mathbb{C}} \bigwedge_{\mathbb{C}}(q)$ for all positive integers p and q , see [1, Proposition 1.6].

4.11. **Lemma.** $D \widehat{\otimes}_{\mathbb{C}} D$ is graded Morita equivalent to the superalgebras \mathbb{C} .

Proof. Set $A = D \widehat{\otimes}_{\mathbb{C}} D$. Let $\varepsilon_{\pm} = \frac{1}{2}(1 \otimes 1 \pm \mathbf{i} \varepsilon \otimes \varepsilon)$, where \mathbf{i} is the imaginary unit of \mathbb{C} . Then $\varepsilon_{\pm}^2 = \varepsilon_{\pm}$, $\varepsilon_+ \varepsilon_- = 0 = \varepsilon_- \varepsilon_+$ and $\varepsilon_+ + \varepsilon_- = 1 \otimes 1$, which implies that $\{\varepsilon_+, \varepsilon_-\}$ is the complete set of orthogonal primitive idempotents of D^2 since $\dim_{\mathbb{C}} A_0 = 2$. It is straightforward to show that

$$\varepsilon_+ A \varepsilon_- = \{c(\mathbf{i} \otimes \varepsilon - 1 \otimes \varepsilon) \mid c \in \mathbb{C}\} \quad \text{and} \quad \varepsilon_- A \varepsilon_+ = \{c(\mathbf{i} \otimes \varepsilon + \varepsilon \otimes 1) \mid c \in \mathbb{C}\}.$$

Let $x = \frac{1}{2}(\mathbf{i} \otimes \varepsilon - 1 \otimes \varepsilon)$ and $y = \frac{1}{2}(\varepsilon \otimes 1 + \mathbf{i} \otimes \varepsilon)$. Then $x \varepsilon_- y = \varepsilon_+$ and $y \varepsilon_+ x = \varepsilon_-$. Applying Lemma 3.7, ε_+ and ε_- are σ -equivalent orthogonal primitive idempotents of A . Thus Proposition-Definition 3.8 implies that A is graded Morita equivalent to $\varepsilon_+ A \varepsilon_+ = \mathbb{C} \varepsilon_+ \cong \mathbb{C}$. \square

Now we yield the graded Morita equivalent classification for all complex Clifford superalgebras.

4.12. Theorem. *Assume that p and q are non-negative integers. Then $\mathbb{C}_{p,q}$ is graded Morita equivalent to graded basic superalgebra $\mathbb{C}_{p(\bmod 2),q}$.*

Proof. Note that $\mathbb{C}_{p,q} \cong \mathbb{C}_{1,0} \widehat{\otimes}_{\mathbb{C}} \bigwedge_{\mathbb{C}}(q)$ and $\mathbb{C}_{1,0} \cong D$. It follows immediately that $\mathbb{C}_{p,q} \approx_{\sigma} \mathbb{C}_{p(\bmod 2),q}$ by using Lemmas 4.11 and 3.10. On the other hand, note that $\widehat{\mathbb{C}_{p,q}} \cong \mathbb{C}_{p,q}$, we have $\mathbb{C}_{p,q} \approx_{\pi} \mathbb{C}_{p(\bmod 2),q}$. We complete the proof. \square

5. THE GROTHENDIECK GROUPS OF CLIFFORD SUPERALGEBRAS

5.1. Most of the important applications of Clifford Superalgebras come through a detailed understanding of their (\mathbb{Z}_2 -graded) representations. This understanding follows rather easily from the graded Morita equivalent classification given by Theorems 4.8 and 4.12. In this section, we shall restrict our attention to the Clifford superalgebras $\mathbb{R}_{p,q}$ and $\mathbb{C}_{p,0}$ for all positive integers p and q in order to simplify our presentation, which are exactly those Clifford (super)algebras play a fundamental role in [1, 8].

5.2. We begin with some notations. For positive integers p and q , let $\text{Irr}_{p,q}$ be the set of all nonequivalent irreducible graded \mathbb{R} -module for $\mathbb{R}_{p,q}$ in the category $\text{Gr } \mathbb{R}_{p,q}$. Similarly, let $\text{Irr}_p^{\mathbb{C}}$ be the set of all nonequivalent irreducible graded \mathbb{R} -module for $\mathbb{C}_{p,0}$ in the category $\text{Gr } \mathbb{C}_{p,0}$.

An object which will be of our interest is the following. Let $K_{p,q}$ be the Grothendieck group of the category of finite dimensional graded real representations of $\mathbb{R}_{p,q}$, and let $K_p^{\mathbb{C}}$ be the Grothendieck group of the category of finite dimensional graded complex representations of $\mathbb{C}_{p,0}$. Then $K_{p,q}$ and $K_p^{\mathbb{C}}$ are the free abelian groups generated by $\text{Irr}_{p,q}$ and $\text{Irr}_p^{\mathbb{C}}$ respectively.

From the classification of Theorems 4.8 and 4.12 we immediately conclude the following:

5.3. Lemma. *Let $v_{p,q}$ and $v_p^{\mathbb{C}}$ denote the cardinality of $\text{Irr}_{p,q}$ and of $\text{Irr}_p^{\mathbb{C}}$ respectively. Then*

$$v_{p,q} = \begin{cases} 2, & \text{if } p - q \equiv 0 \pmod{4} \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad v_p^{\mathbb{C}} = \begin{cases} 2, & \text{if } p \text{ is odd} \\ 1, & \text{if } p \text{ is even.} \end{cases}$$

Now we can obtain the main result of this section.

5.4. Theorem. *Let p and q be positive integers.*

(i) *The elements $\text{Irr}_{p,q}$, $v_{p,q}$ and $K_{p,q}$ are as given in the following table:*

$p - q \pmod{8}$	0	1	2	3	4	5	6	7
$\mathbb{R}_{p,q}/\approx$	\mathbb{R}	$\mathbb{R}_{1,0}$	$\mathbb{R}_{2,0}$	$\mathbb{R}_{3,0}$	\mathbb{H}	$\mathbb{R}_{0,1}$	$\mathbb{R}_{0,2}$	$\mathbb{R}_{0,3}$
$\text{Irr}_{p,q}/\approx$	$\{\mathbb{R}, \sigma(\mathbb{R})\}$	$\{\mathbb{R}_{1,0}\}$	$\{\mathbb{R}_{2,0}\}$	$\{\mathbb{R}_{3,0}\}$	$\{\mathbb{H}, \sigma(\mathbb{H})\}$	$\{\mathbb{R}_{0,1}\}$	$\{\mathbb{R}_{0,2}\}$	$\{\mathbb{R}_{0,3}\}$
$v_{p,q}$	2	1	1	1	2	1	1	1
$K_{p,q}$	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}

(ii) *$v_p^{\mathbb{C}}$, $\text{Irr}_p^{\mathbb{C}}$ and $K_p^{\mathbb{C}}$ are given by following table:*

$p \pmod{2}$	\mathbb{C}_p/\approx	$\text{Irr}_p^{\mathbb{C}}/\approx$	$v_p^{\mathbb{C}}$	$K_p^{\mathbb{C}}$
0	\mathbb{C}	$\{\mathbb{C}, \sigma(\mathbb{C})\}$	2	$\mathbb{Z} \oplus \mathbb{Z}$
1	$\mathbb{C}_{1,0}$	$\{\mathbb{C}_{1,0}\}$	1	\mathbb{Z}

Proof. Note that the graded modules of a gr-divisional superalgebra without non-trivial odd part are some copies of this gr-divisional superalgebra. Then the theorem is a direct consequence of Theorems 4.8 and 4.12 and Lemma 5.3. \square

We remark in closing that Atiyah, Bott and Shapiro obtained Theorem 5.4 by using the periodicity theorem for Clifford superalgebras [1, § 5], and it is important to understand the Atiyah-Bott-Shapiro isomorphisms [1, Theorem 11.5], see [1] and [8] for more details.

ACKNOWLEDGEMENTS

The note develops partly form a part of [14]. The author is very grateful to Professors Yang Han and Yingbo Zhang for their invaluable help. The author would like to thank the referees for their comments and suggestions, which contributed to improvements in the presentation of the results.

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